

A CONSISTENCY PROOF FOR SOME RESTRICTIONS OF TAIT'S REFLECTION PRINCIPLES

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ABSTRACT. In [4] Tait identifies a set of reflection principles which he calls $\Gamma_n^{(2)}$ -reflection principles which Peter Koellner has shown to be consistent relative to $\kappa(\omega)$, the first ω -Erdős cardinal, in [2]. Tait also goes on in the same work to define a set of reflection principles which he calls $\Gamma_n^{(m)}$ -reflection principles; however Koellner has shown that these are inconsistent when $m > 2$ in [3], but identifies restricted versions of them which he proves consistent relative to $\kappa(\omega)$. In this paper we introduce a new large-cardinal property with an ordinal parameter α , calling those cardinals which satisfy it α -reflective cardinals. Its definition is motivated by the remarks Tait makes in [4] about why reflection principles must be restricted when parameters of third or higher order are introduced. We prove that if κ is $(\alpha + 1)$ -strong and $\alpha < \kappa$ then κ is α -reflective. Furthermore we show that α -reflective cardinals relativize to L , and that if $\kappa(\omega)$ exists then the set of cardinals $\lambda < \kappa(\omega)$ such that λ is α -reflective for all $\alpha < \lambda$. We show that an ω -reflective cardinal satisfies some restricted versions of $\Gamma_n^{(m)}$ -reflection, as well as all the reflection properties which Koellner proves consistent in [3].

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1. INTRODUCTION

We are going to investigate reflection principles, which postulate the existence of a level of the universe V_κ , whose properties reflect down to some lower level V_β where $\beta < \kappa$. It is useful to begin by considering reflection principles involving second-order parameters only. In later sections we will consider the issues which arise when one introduces higher-order parameters.

The cardinals yielded by these reflection principles involving second-order parameters only are called “indescribable cardinals”. These principles assert the existence of a cardinal κ such that certain statements true in V_κ hold when relativized to a level V_β where $\beta < \kappa$. The strength of the reflection principles increase as one increases the expressive power of the language in which the statements are formulated, and the complexity of the formulas which express them. For example, one may consider the case where the language L in which the statements are expressed is the first-order language of set theory extended by variables of all finite orders. We denote the order of a variable with a superscript, so that $X^{(m)}$ is a variable of m th order. Second-order variables range over classes, for example. If a formula φ in the language L is relativized to V_κ , then the variables of m th order range over $V_{\kappa+m-1}$. Here V_κ is being treated as the universe.

Definition 1.1. We say that a formula in the language \mathcal{L} is a Π_0^m -formula if the only quantified variables it contains are at most m th order.

We say that a formula in the language \mathcal{L} is a Π_1^m -formula if it consists of a block of universal $(m+1)$ th order quantifiers tacked on to the beginning of a Π_0^m -formula.

We say that a formula in the language \mathcal{L} is Σ_{k+1}^m if it consists of a block of existential $(m+1)$ th-order quantifiers tacked on to the beginning of a Π_k^m -formula.

We say that a formula in the language \mathcal{L} is Π_{k+1}^m if it consists of a block of universal $(m+1)$ th-order quantifiers tacked on to the beginning of a Σ_k^m -formula.

Definition 1.2. If φ is formula in the language \mathcal{L} , we denote by φ^β the result of relativizing every m th-order quantifier to $V_{\beta+m-1}$. If $X^{(2)}$ is a second-order variable we abbreviate $X^{(2)} \cap V_\beta$ to $X^{(2),\beta}$.

Definition 1.3. If Ω is a class of formulas, we say that κ is Ω -indescribable if for all formulas $\varphi \in \Omega$ whose only free variable is second-order, for all sets $U \subset V_\kappa$, $\varphi^\kappa(U) \implies \exists \beta < \kappa \varphi^\beta(U^\beta)$. We say that κ is totally indescribable if it is Π_n^m -indescribable for all $m, n > 0$.

Definition 1.4. Suppose that α is an ordinal. We say that κ is α -indescribable if for all Π_0^1 formulas φ in the language \mathcal{L} whose only free variable is second-order, for all sets $U \subset V_\kappa$, $V_{\kappa+\alpha} \models \varphi(U) \implies \exists \beta < \kappa V_{\beta+\alpha} \models \varphi(U^\beta)$ for some $\beta < \kappa$.

Definition 1.5. We say that κ is absolutely indescribable if κ is α -indescribable for all $\alpha < \kappa$.

Definition 1.6. We say that κ is extremely indescribable if for all formulas Π_0^1 formulas φ in the language \mathcal{L} whose only free variable is second-order, for all sets $U \subset V_\kappa$, $V_{\kappa+\kappa} \models \varphi(U) \implies \exists \beta < \kappa V_{\beta+\beta} \models \varphi(U^\beta)$.

Here we are giving examples of cardinals κ such that V_κ satisfies reflection of formulas with second-order parameters. Let us next consider what happens when we move to parameters of third or higher order.

2. REFLECTION INVOLVING PARAMETERS OF THIRD OR HIGHER ORDER

We have already defined $A^{(2),\beta}$ when $A^{(2)}$ is a second-order parameter. We define $A^{(m+1),\beta} = \{B^{(m),\beta} \mid B^{(m)} \in A^{(m+1)}\}$ for all integers $m \geq 2$. We say that κ satisfies reflection with m th-order parameters for all formulas in a class Ω if, whenever $\varphi^\kappa(U^{(m)})$ for some $U^{(m)} \subset V_{\kappa+m-1}$, there exists a $\beta < \kappa$ such that $\varphi^\beta(U^{(m),\beta})$. It is inconsistent to postulate the existence of cardinal κ which satisfies reflection for all formulas in \mathcal{L} , where \mathcal{L} is the language defined in the second paragraph of the Introduction, with third-order parameters. To see this, let $A^{(3)}$ be a third-order parameter and let φ be the assertion that every element of $A^{(3)}$ is a bounded subset of On . This assertion can be written as a sentence in \mathcal{L} with a third-order parameter. Now, suppose that κ satisfies reflection for such sentences with third-order parameters. Let $U^{(3)} = \{\{\xi \mid \xi < \alpha\} \mid \alpha \in \text{On} \cap \kappa\}$. We have $\varphi^\kappa(U^{(3)})$. So by the hypothesis about κ we must have $\varphi^\beta(U^{(3),\beta})$ for some $\beta < \kappa$. But this is impossible because $U^{(3),\beta}$ contains the set $\{\xi \mid \xi < \beta\}$, which is not bounded in $\text{On} \cap V_\beta$. Thus no ordinal κ satisfies reflection for formulas in \mathcal{L} with third-order parameters.

This means that in order to formulate consistent reflection principles for formulas with third-order parameters or higher one must constrain the formulas relativized in some way. Let us consider what Tait writes in [4] about this issue.

“One plausible way to think about the difference between reflecting $\varphi(A)$ when A is second-order and when it is of higher-order is that, in the former case, reflection is asserting that, if $\varphi(A)$ holds in the structure $\langle R(\kappa), \in, A \rangle$, then it holds in the substructure $\langle R(\beta), \in, A^\beta \rangle$ for some $\beta < \kappa \dots$ But, when A is higher-order, say of third-order this is no longer so. Now we are considering the structure $\langle R(\kappa), R(\kappa + 1), \in, A \rangle$ and $\langle R(\beta), R(\beta + 1), \in, A^\beta \rangle$. But, the latter is not a substructure of the former, that is the ‘inclusion map’ of the latter structure into the former is no longer single-valued: for subclasses X and Y of $R(\kappa)$, $X \neq Y$ does not imply $X^\beta \neq Y^\beta$. Likewise for $X \in R(\beta + 1)$, $X \notin A$ does not imply $X^\beta \notin A^\beta$. For this reason, the formulas that we can expect to be preserved in passing from the former structure to the latter must be suitably restricted and, in particular, should not contain the relation \notin between second- and third-order objects or the relation \neq between second-order objects.”

Now, suppose that we are reflecting a formula φ of the form

$$\forall X_1^{(m_1)} \exists Y_1^{(n_1)} \forall X_2^{(m_2)} \exists Y_2^{(n_2)} \dots \forall X_k^{(m_k)} \exists Y_k^{(n_k)} \\ \psi(X_1^{(m_1)}, Y_1^{(n_1)}, X_2^{(m_2)}, Y_2^{(n_2)}, \dots, X_k^{(m_k)}, Y_k^{(n_k)}, A_1^{(l_1)}, A_2^{(l_2)}, \dots, A_j^{(l_j)})$$

This can be re-written as

$$\exists f_1 \exists f_2 \dots \exists f_k \forall X_1^{(m_1)} \forall X_2^{(m_2)} \dots \forall X_k^{(m_k)} \\ \psi(X_1^{(m_1)}, f_1(X_1^{(m_1)}), X_2^{(m_2)}, f_2(X_1^{(m_1)}, X_2^{(m_2)}), \dots, X_k^{(m_k)}, f_k(X_1^{(m_1)}, X_2^{(m_2)}, \dots, X_k^{(m_k)}), \\ A_1^{(l_1)}, A_2^{(l_2)}, \dots, A_j^{(l_j)})$$

The point is that if this formula, without the existential function quantifiers, is conceived of as holding in the structure $\langle V_\kappa, V_{\kappa+1}, \dots, V_{\kappa+l}, \in, f_1, \dots, f_k, A_1^{(l_1)}, A_2^{(l_2)}, \dots, A_j^{(l_j)} \rangle$, where $l = \max(m_1, n_1, \dots, m_k, n_k, l_1 - 1, \dots, l_j - 1) - 1$, and we try to reflect down to the structure $\langle V_\beta, V_{\beta+1}, \dots, V_{\beta+l}, \in, f_1^\beta, \dots, f_k^\beta, A_1^{(l_1),\beta}, A_2^{(l_2),\beta}, \dots, A_j^{(l_j),\beta} \rangle$ for some $\beta < \kappa$, then the functions f_i^β are no longer necessarily single-valued. This consideration suggests the following reflection principle.

Definition 2.1. We define $l(\gamma) = \gamma - 1$ if $\gamma < \omega$ and $l(\gamma) = \gamma$ otherwise. We extend the definition $A^{(m+1),\beta} = \{B^{(m),\beta} \mid B^{(m)} \in A^{(m+1)}\}$ to $A^{(\alpha),\beta} = \{B^\beta \mid B \in A^{(\alpha)}\}$ for all ordinals $\alpha > 1$, it being understood that if V_κ is the domain of discourse then $A^{(\alpha)}$ ranges over $V_{\kappa+l(\alpha)}$.

Definition 2.2. Suppose that α, κ are ordinals such that $\alpha < \kappa$ and that

(1) $S = \langle \{V_{\kappa+\gamma} \mid \gamma < \alpha\}, \in, f_1, f_2, \dots, f_k, A_1, A_2, \dots, A_n \rangle$ is a structure where each f_i is a function $V_{\kappa+l(\gamma_1)} \times V_{\kappa+l(\gamma_2)} \times \dots \times V_{\kappa+l(\gamma_i)} \rightarrow V_{\kappa+\zeta_i}$ for some ordinals $\gamma_1, \gamma_2, \dots, \gamma_i, \zeta_i$ such that $l(\gamma_1), l(\gamma_2), \dots, l(\gamma_i), \zeta_i < \alpha$, and each A_i is a subset of $V_{\kappa+l(\delta_i)}$ for some $\delta_i < \alpha$

(2) φ is a formula true in the structure S , of the form

$$\forall X_1^{(\gamma_1)} \forall X_2^{(\gamma_2)} \dots \forall X_k^{(\gamma_k)} \\ \psi(X_1^{(\gamma_1)}, f_1(X_1^{(\gamma_1)}), X_2^{(\gamma_2)}, f_2(X_1^{(\gamma_1)}, X_2^{(\gamma_2)}), \dots, X_k^{(\gamma_k)}, f_k(X_1^{(\gamma_1)}, X_2^{(\gamma_2)}, \dots, X_k^{(\gamma_k)}), \\ A_1, A_2, \dots, A_j) \text{ with } \psi \text{ a formula with first-order quantifiers only}$$

(3) there exists a β such that $\alpha < \beta < \kappa$ and a mapping $j : V_{\beta+\alpha} \rightarrow V_{\kappa+\alpha}$, such that $j(X) \in V_{\kappa+\gamma}$ whenever $X \in V_{\beta+\gamma}$, $j(X) = X$ for all $X \in V_\beta$, and $j(X) \in j(Y)$ whenever $X \in Y$, and such that, in the structure

$S^\beta = \langle V_\beta, \{V_{\beta+\gamma} \mid 0 < \gamma < \alpha\}, \{V_{\kappa+\gamma} \mid 0 < \gamma < \alpha\}, \in, j, f_1, f_2, \dots, f_k, A_1, A_2, \dots, A_n \rangle$, with variables of order γ ranging over $V_{\beta+l(\gamma)}$, we have

$$\forall X_1^{(\gamma_1)} \forall X_2^{(\gamma_2)} \dots \forall X_k^{(\gamma_k)} \\ \psi(j(X_1^{(\gamma_1)}), f_1(j(X_1^{(\gamma_1)})), j(X_2^{(\gamma_2)}), f_2(j(X_1^{(\gamma_1)}), j(X_2^{(\gamma_2)})), \dots, j(X_k^{(\gamma_k)}), f_k(j(X_1^{(\gamma_1)}), j(X_2^{(\gamma_2)}), \dots, j(X_k^{(\gamma_k)})), A_1, A_2, \dots, A_n)$$

Then we say that the formula φ with parameters A_1, A_2, \dots, A_n reflects down from S to β . If for all structures S of the above form and for all formulas φ of the above form true in the structure S , this occurs for some $\beta < \kappa$, then κ is said to be α -reflective.

It is not clear whether it should be said that the existence of α -reflective cardinals follows from the iterative conception of set, because the definition involves a function j which guides the reflection. The idea that the existence of indescribable cardinals follows from the iterative conception of set can be motivated by an idea of Tait [4] which Koellner has called the Relativised Cantorian Principle. Cantor wrote that if an initial segment of the sequence of ordinals is only a set then it has a least strict upper bound. The phrase “is only a set” can be replaced with other conditions for the existence of a least strict upper bound, and for any given set of conditions it then becomes plausible to postulate the existence of a level of the universe which is a closure point for the process of obtaining new ordinals in this way. The indescribable cardinals can then be motivated by the idea that if a level of the universe is describable then it cannot be all of V , and so this is a reasonable condition for the existence of a least strict upper bound of all the ordinals obtained so far, and it is reasonable to postulate the existence of a level of the universe which is a closure point for the process of obtaining new ordinals in this way. So whether or not one should similarly admit the existence of reflective cardinals as defined above depends on whether or not one thinks it reasonable for the nonexistence of a function j guiding the reflection of the formula is a sufficient reason to think that the level of the universe obtained so far is not all of V , and whether it is reasonable to postulate the existence of a level of the universe which is a closure point for the process of obtaining new ordinals in this way. This may seem doubtful. The Relativised Cantorian Principle always runs the risk of proving too much.

There is however another way to motivate a justification for these cardinals. In [1], Hellman discusses the notion of a level of the universe V_α “Putnam-satisfying” a higher-order formula with parameters. The levels V_α with α inaccessible all agree on what first-order formulas they Putnam-satisfy, but not for the higher-order formulas. If one postulates a reflection principle whereby if a level of the universe satisfies a higher-order formula with parameters then a lower level Putnam-satisfies it with the same parameters, then one can then proceed to prove the existence (assuming the axiom of choice) of the reflective cardinals discussed here. This might be thought to be a somewhat more compelling justification.

At any rate I hope that the results below are of mathematical interest and shed some further light on the notion of a reflection principle and when we should expect reflection principles to be consistent.

We now give a consistency proof for this large cardinal property.

Theorem 2.3. *Suppose that $\alpha < \kappa$ and κ is $\alpha + 1$ -strong. Then κ is α -reflective.*

Proof. Suppose that $\alpha < \kappa$ and κ is $\alpha + 1$ -strong. Then there exists an elementary embedding $k : V \rightarrow M$ with critical point κ such that $V_{\kappa+\alpha+1} \subset M$. Let $S = \langle \{V_{\kappa+\gamma} \mid \gamma < \alpha\}, \in, f_1, f_2, \dots, f_k, A_1, A_2, \dots, A_n \rangle$ be a structure and φ a formula as in the definition of an α -reflective cardinal. Working in M , consider the structure $k(S)$. Since $V_{\kappa+\alpha+1} \subset M$, the elementary embedding k induces a mapping $j \in M$ as in the definition of an α -reflective cardinal such that the structure $k(S)$ reflects down to κ in M via the mapping j . Since k is an elementary embedding we may infer that there exists a $\delta < \kappa$ such that S reflects down to δ in V . \square

Next we show that α -reflective cardinals relativise to the constructible universe L .

Theorem 2.4. *Suppose that $\alpha < \kappa$ and κ is α -reflective. Then κ is α -reflective in the constructible universe L .*

Proof. Suppose that $\alpha < \kappa$. Let $S = \langle \{V_{\kappa+\gamma} \mid \gamma < \alpha\}, \in, f_1, f_2, \dots, f_k, A_1, A_2, \dots, A_n \rangle$ be a structure as in the definition of an α -reflective cardinal, with all the functions and predicates being members of L . Let $S^L = \langle \{V_{\kappa+\gamma}^L \mid \gamma < \alpha\}, \in, f_1, f_2, \dots, f_k, A_1, A_2, \dots, A_n \rangle \in L$ and φ be a formula as in the definition of an α -reflective cardinal which is true in S^L . We may consider the formula φ^L with all γ -order quantifiers for $\gamma < \alpha$ relativized to $V_{\kappa+l(\gamma)}^L$. This is a formula which is true in S . By introducing new Skolem functions for φ^L into the structure S to produce an expanded structure S' , we may replace φ^L with a formula ψ which is true in the expanded structure S' . Then since κ is α -reflective in V then there must be a mapping j which witnesses that ψ reflects down to some $\beta < \kappa$. One can ensure that $j \restriction_L \in L$ by defining j by means of the canonical well-ordering of L . This shows that φ reflects down from S^L to β in L . \square

Next we show that these cardinals are consistent relative to $\kappa(\omega)$.

Theorem 2.5. *Suppose that $\kappa(\omega)$ exists. Then the set of all $\lambda < \kappa(\omega)$ such that λ is α -reflective for all $\alpha < \lambda$ is a stationary subset of $\kappa(\omega)$.*

Proof. Suppose that $\kappa = \kappa(\omega)$. Let C be a closed bounded subset of κ . We must show that there is a cardinal $\lambda \in C$ such that λ is α -reflective for all $\alpha < \lambda$. Let $S = \{\iota_1, \iota_2, \dots\}$ be a set of Silver indiscernibles for the structure $\langle V_\kappa, \epsilon, C \rangle$. It can be shown that all the ι_i are in C . Let M be a Skolem hull of S in this structure and let $\lambda = \iota_2$. Then the mapping $\iota_k \mapsto \iota_{k+1}$ induces an elementary embedding $j : M \rightarrow M$. If we let $M_{n,n'}$ be the Skolem hull of $(V_\lambda)^M \cup \{\iota_1, \iota_2, \dots, \iota_{n'}\}$ in M for Skolem terms for formulas of complexity no greater than Σ_n , then the mapping $\iota_k \mapsto \iota_{k+1}$ together with a well-ordering of $(V_\lambda)^M$ induces a mapping $j_{n,n'} : M_{n,n'} \rightarrow M_{n,n'+1}$ which respects the Skolem functions for formulas of complexity no greater than Σ_n , and truncates of this mapping are members of M . If φ is a formula with parameters A_1, A_2, \dots, A_m as in the definition of an α -reflective cardinal where $\alpha < \lambda$ then φ will reflect down in M from ι_2 to ι_1 by means of a truncation of the mapping $j_{n,n'}$ where n and n' are sufficiently large. This shows that ι_2 is α -reflective in M and hence in V . \square

Next we establish some properties of ω -reflective cardinals.

3. RESTRICTED VERSIONS OF TAIT'S REFLECTION PRINCIPLES

In [4] Tait defines the following set of reflection principles.

Definition 3.1. A formula in the language of finite orders is positive iff it is built up by means of the operations $\vee, \wedge, \forall, \exists$ from atoms of the form $x = y, x \neq y, x \in y, x \notin y, x \in Y^{(2)}, x \notin Y^{(2)}$ and $X^{(m)} = X'^{(m)}$ and $X^{(m)} \in Y^{(m+1)}$, where $m \geq 2$.

Definition 3.2. For $0 < n < \omega$, $\Gamma_n^{(2)}$ is the class of formulas

$$(1) \quad \forall X_1^{(2)} \exists Y_1^{(k_1)} \dots \forall X_n^{(2)} \exists Y_n^{(k_n)} \varphi(X_1^{(2)}, Y_1^{(k_1)}, \dots, X_n^{(2)}, Y_n^{(k_n)}, A^{(l_1)}, \dots, A^{(l_{n'})})$$

where φ does not have quantifiers or second or higher-order and $k_1, \dots, k_n, l_1, \dots, l_{n'}$ are natural numbers.

Definition 3.3. We say that V_α satisfies $\Gamma_n^{(2)}$ -reflection if for each formula $\varphi \in \Gamma_n^{(2)}$, if $V_\alpha \models \varphi$ then there is a $\delta < \alpha$ such that $V_\alpha \models \varphi^\delta$.

Theorem 3.4 (Koellner). Suppose that $\kappa = \kappa(\omega)$ is the first ω -Erdős cardinal. Then there exists a $\delta < \kappa$ such that V_δ satisfies $\Gamma_n^{(2)}$ -reflection for all n .

Theorem 3.5 (Tait). Suppose that $n < \omega$ and V_κ satisfies $\Gamma_n^{(2)}$ -reflection. Then κ is n -ineffable.

Theorem 3.6 (Tait). Suppose that κ is measurable. Then V_κ satisfies $\Gamma_n^{(2)}$ -reflection for all $n < \omega$.

In [4] Tait proposes to define $\Gamma_n^{(m)}$ in the same way as the class of formulas $\Gamma_n^{(2)}$, except that universal quantifiers of order $\leq m$ are permitted. Koellner shows in [3] that this form of reflection is inconsistent when $m > 2$. We formulate a new form of reflection which we will be able to prove holds for an ω -reflecting cardinal.

Definition 3.7. For $2 \leq m < \omega$, $0 < n < \omega$, $\Gamma_n^{*(m)}$ is the class of formulas

$$(2) \quad \forall X_1^{(k_1)} \exists Y_1^{(l_1)} \dots \forall X_n^{(k_n)} \exists Y_n^{(l_n)} \psi(X_1^{(k_1)}, Y_1^{(l_1)}, \dots, X_n^{(k_n)}, Y_n^{(l_n)}, A^{(m_1)}, \dots, A^{(m_p)})$$

where ψ does not have quantifiers or second or higher-order and $k_1, \dots, k_n, l_1, \dots, l_n, m_1, \dots, m_p$ are natural numbers such that $l_i \geq k_j$ whenever $0 < i \leq j \leq n$.

Definition 3.8. We say that V_κ satisfies $\Gamma_n^{*(m)}$ -reflection if, for all $\varphi \in \Gamma_n^{*(m)}$, if $V_\kappa \models \varphi(A^{(m_1)}, A^{(m_2)}, \dots, A^{(m_p)})$ then $V_\kappa \models \varphi^\delta(A^{(m_1),\delta}, A^{(m_2),\delta}, \dots, A^{(m_p),\delta})$ for some $\delta < \kappa$.

We shall now prove that if κ is ω -reflective then V_κ satisfies $\Gamma_n^{*(m)}$ -reflection for all $m \geq 2, n > 0$. Note that $\Gamma_n^{*(2)}$ -reflection is the same as $\Gamma_n^{(2)}$ -reflection.

Theorem 3.9. Suppose that κ is ω -reflective. Then V_κ satisfies $\Gamma_n^{*(m)}$ -reflection for all $m \geq 2, n > 0$.

Proof. Suppose that $\varphi \in \Gamma_n^{*(m)}$ is true in V_κ and that φ is as in Formula 2. There must exist functions f_1, f_2, \dots, f_n such that

$$(3) \quad \forall X_1^{(k_1)} \dots \forall X_n^{(k_n)} \psi(X_1^{(k_1)}, f_1(X_1^{(k_1)}), \dots, X_n^{(k_n)}, f_n(X_1^{(k_1)}, X_2^{(k_2)}, \dots, X_n^{(k_n)}), A^{(m_1)}, \dots, A^{(m_p)})$$

is true in V_κ . Since κ is ω -reflective there will be some $\beta < \kappa$ and a function $j : V_{\beta+\omega} \rightarrow V_{\kappa+\omega}$ as in the definition of an ω -reflective cardinal such that

$$(4) \quad \forall X_1^{(k_1)} \dots \forall X_n^{(k_n)} \psi(j(X_1^{(k_1)}), f_1(j(X_1^{(k_1)})), \dots, j(X_n^{(k_n)}), f_n(j(X_1^{(k_1)}), j(X_2^{(k_2)}), \dots, j(X_n^{(k_n)})), A^{(m_1)}, \dots, A^{(m_p)})$$

is true in V_β . As Koellner observes in [2], when $k_i = 2$ for each i this is enough to prove $\Gamma_n^{(2)}$ -reflection because the map $X^{(2)} \mapsto j(X^{(2)}) \cap V_\beta$ is surjective on $V_{\beta+1}$. To establish $\Gamma_n^{*(m)}$ -reflection for $m > 2$, we replace j in the above formula with the function j' which agrees with j on $V_{\beta+1}$, and satisfies $j'(X) = \{j'(Y) \mid Y \in X\}$ on $V_{\beta+k} \setminus V_{\beta+k-1}$, for $k = 2, \dots, m$. The part of the formula inside the quantifiers will certainly remain true in V_κ . Since there exists a function k such that $j = k \circ j'$, the formula will remain true in V_β as it is equivalent to a formula asserting the existence of certain Skolem functions pickinng out appropriate values for the first-order variables, and the Skolem functions which witness the truth of the formula with j replaced wiht j' can be composed on the right with k . This completes the proof. \square

It is also easy to see by examining Koellner's proofs in [3] that ω -reflective cardinals satisfy the reflection principles which he proves consistent there.

It is plausible to regard α -reflective cardinals as the natural generalization of Tait's proposed reflection principles. The fact that they do not break the $\kappa(\omega)$ barrier provides further evidence for the view that Koellner has expressed in [3] that no reflection principle does so, and reflection principles are not sufficient to effect a significant reduction in incompleteness of ZFC.

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